

# Big, Fast Vortices in the d-RVB theory of high temperature superconductivity

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The effect of proximity to a Mott insulating phase on the superflow properties of a d-wave superconductor is studied using the slave boson- $U(1)$  gauge theory model. The model has two limits corresponding to superconductivity emerging either out of a 'renormalized fermi liquid' or out of a non-fermi-liquid regime. Three crucial physical parameters are identified: the size of the vortex *as determined from the supercurrent it induces*; the coupling of the superflow to the quasiparticles and the 'nondissipative time derivative' term. As the Mott phase is approached, the core size as defined from the supercurrent diverges, the coupling between superflow and quasiparticles vanishes, and the magnitude of the nondissipative time derivative dramatically increases. The dissipation due to a moving vortex is found to vary as the third power of the doping. The upper critical field and the size of the critical regime in which paraconductivity may be observed are estimated, and found to be controlled by the supercurrent length scale.

## INTRODUCTION

High- $T_c$  superconductors are created by doping an antiferromagnetic 'Mott insulating' parent material, and the effect of proximity to the Mott phase on their superconducting properties remains a crucial and still incompletely understood issue [1]. One expects on general grounds that the suppression of current response near a Mott insulator leads to 'type II' behavior, so a fundamental issue is the physics associated with vortices in the superconducting order parameter. An isolated vortex involves a quantized flux ( $hc/2e$  in conventional superconductors), a circulating supercurrent pattern and a 'core region' in which the quasiparticle excitation spectrum differs from that observed far from the vortex. The possibility (apparently not realized in known superconductors) that proximity to the Mott phase could induce an unconventional value of the flux quantum has been discussed [2, 3]. An extensive literature exists on quasiparticle properties (including the possibility of interesting discrete core states [4], and whether an antiferromagnetic [5, 6] or other [7] state is induced in or near the vortex core). However, apart from the pioneering phenomenological work of Lee and Wen [8] and an analysis of the resistive transition in overdoped  $Tl$ -based materials [9], little theoretical attention has been paid to the superflow properties even though these in fact control many physically important quantities including  $H_{c2}$  and the size of the 'critical regime' in which superconducting fluctuation properties may be observed in the conductivity.

Theoretical analysis of vortex properties requires a model. Conventional models of superconductivity in interacting electron systems are based on Landau's fermi

liquid theory, but as we show in the Appendix, analysis of the changes occurring as the Mott phase is approached requires a model which goes beyond Fermi liquid theory, at minimum by including effects corresponding to a scale dependence of a Landau parameter and perhaps more fundamentally by allowing for superconductivity to emerge from a fundamentally non-fermi-liquid state or regime. One widely studied theoretical model of a doped Mott insulator is the  $U(1)$  gauge theory implementation [10] of the RVB ideas of P. W. Anderson [11]. This theory and its variants have been extensively studied as an approximation to the low energy physics of the  $t - J$  model believed [1] to capture the essential aspects of the low energy physics of high- $T_c$  materials. It exhibits (at least in a large- $N$  limit) a non-fermi-liquid regime [10, 12] involving exotic excitations ('spinons' and 'holons' coupled by a gauge field) and a fermi liquid regime in which the spinon and holon are bound together into a conventional electron and the gauge field effects produce a nontrivial doping dependence of the Landau parameter  $F_{1S}$  [13, 14]. The model also possesses a d-wave superconducting state [15] which may emerge either from the fermi-liquid or non-fermi-liquid regimes. Quasiparticle properties (including a possible antiferromagnetic [6] or staggered flux [7] state in the core of the vortex) have been studied and the model has been shown to admit  $h/2e$  vortices [16], but superfluid properties such as the supercurrent distribution in the vortex state and the dissipation occurring when a vortex moves have been less well studied.

As discussed at length elsewhere [8, 14, 18] this theory disagrees with experiment in a number of ways. Most problematically, the model predicts a strong doping dependence to the leading low- $T$  correction to the London

penetration depth ( $d\lambda^{-2}/dT \sim (\text{doping})^2$ ) which is not observed. We therefore do not believe the theory is a realistic representation of high temperature superconductors; however it is a very useful model system. We stress that although as usually formulated the model involves exotic excitations such as holons and spinons, for the properties we discuss these can be completely eliminated: as shown in the text and Appendix, the model can be viewed simply as a method of calculating the behavior of a fermi-liquid based system at length scales short enough that scale dependence of the Landau parameters becomes important.

The importance of the model is that it provides an explicit realization of a situation (which, we believe, is generically realized in lightly doped Mott insulators) where the supercurrent-defined and quasiparticle-defined length scales are parametrically different. We show in this paper that the longer length scale is in fact the one relevant to the conventional superfluid properties such as  $T_c(H)$  and the size of the 'fluctuation regime' in which the model exhibits (for example) a nontrivial paraconductivity.

The rest of this paper is organized as follows. Section II reviews necessary aspects of the  $U(1)$  gauge theory formalism. Section III discusses in detail the current distribution around a vortex. Section IV calculates the dissipation induced when a vortex moves and uses this information to estimate the size of the 'critical regime' in which superconducting fluctuation effects are visible in the conductivity. Section V considers the low temperature limit of the upper critical field. Section VI is a conclusion, summarizing the results and their implications. An Appendix explicates the relation between the results presented here and the conventional fermi liquid analysis.

## FORMALISM

This subsection reviews results obtained in the early days of the gauge theory [10, 12, 15, 16], in order to establish notation and introduce important concepts.

In the gauge theory one writes the electron  $c_{i\alpha}$  in terms of a charge-e boson  $b$  (representing a hole) and a fermionic 'spinon' representing a spin degree of freedom, thus  $c_{i\sigma}^+ = b_i f_{i\sigma}^+$ . The superconducting state is described by a d-wave BCS pairing of spinons [15] (involving a d-symmetry pairing gap with maximum value  $\Delta$ ) and a condensation of the bosons. The low energy, long-wavelength physics is controlled by the Hamiltonian

$$H_{gauge} = \frac{1}{2}\rho_B(\nabla\phi_B - a - A)^2 + \frac{1}{2}\rho_F(\frac{1}{2}\nabla\phi_F - a)^2 \quad (\#) \\ + H_D + H_{mix} + \dots$$

Here  $a$  is an internal gauge field which enforces the constraint, arising because the physical fermion  $c_i = f_i b_i^+$ ,

that a longitudinal spinon current must cause an equal and opposite boson current. (It is possible to have transverse currents of spinons with no holon motion but these are not relevant here).  $\phi_B$  is the phase of the boson field and  $\rho_B$  is the  $T = 0$  boson superfluid stiffness,  $\phi_F$  is the phase of the spinon (fermion) pairing amplitude and  $\rho_F$  is the corresponding  $T = 0$  spinon 'superfluid' stiffness.  $H_D$  is the usual (normal-ordered) dirac Hamiltonian describing the quasiparticle part of the spinon degrees of freedom and the ellipsis expresses terms irrelevant to the present discussion.

$H_D$  has eigenvalues  $E_p = \sqrt{v_1^2 p_1^2 + v_2^2 p_2^2}$  with  $v_1$  the spinon fermi velocity and  $v_2$  related to the d-wave gap in the usual way. The spinons are coupled to the gauge field and thus to the 'holons' via the term

$$H_{mix} = \sum_{\alpha,\sigma} \left( i\frac{1}{2}\partial_\mu\phi_F - ea_\mu \right) \cdot \vec{v}_1 c_{\alpha\sigma}^+ c_{\alpha\sigma} \quad (2)$$

We begin our analysis of  $H_{gauge}$  by considering length and energy scales. The fermionic part of the Hamiltonian involves the length scale

$$\xi_F = \frac{v_1}{\Delta} \quad (3)$$

and two energy scales:  $\Delta$  and

$$\rho_F \sim v_1 p_F \quad (4)$$

$\xi_F$  is relatively short and does not diverge as the Mott phase is approached, and  $\rho_F$  is relatively large (of order  $J$  in the  $t - J$  model) and does not vanish as the Mott phase is approached.

The boson stiffness  $\rho_B$  has dimension of energy (in two spatial dimensions) and is proportional to the doping  $x$  and to the basic electronic hopping parameter  $t$ .  $\rho_B$  is expected to vary [19] on the length scale  $x^{-1/2}$  which is the distance between charge carriers. We shall be interested primarily in the limit  $x^{-1/2} > \xi_F$ . We note that as the doping is increased,  $\Delta$  decreases [15] and eventually becomes smaller than an energy of order  $x^{1/2}J$  so the inequality is reversed. For larger dopings the theory becomes essentially the familiar *BCS* one, with only one important length scale,  $\xi_F$ .

The currents carried by boson and fermion degrees of freedom are, respectively

$$j_B = \rho_B * (\nabla\phi_B - a - A) \quad (5)$$

$$j_F = \rho_F(\frac{1}{2}\nabla\phi_F - a) \quad (6)$$

Here the  $*$  denotes convolution and is to remind the reader that  $\rho_B$  is scale dependent on scales relevant to the subsequent discussion.

The physical current  $j_{phys} = j_B$  and the constraint enforced by the gauge field  $a$  is  $j_B + j_F = 0$ , i.e.

$$\rho_B * (\nabla\phi_B - a - A) + \rho_F(\frac{1}{2}\nabla\phi_F - a) = 0 \quad (7)$$

This implies

$$a = (\rho_B + \rho_F)^{-1} * (\rho_B * (\nabla\phi_B - A) + \rho_F \nabla\phi_F) \quad (8)$$

In the long wavelength limit the non-locality of  $\rho_B$  may be neglected. As  $T \rightarrow 0$  and assuming no fermions are excited, elimination of  $a$  leads to [10, 16]

$$H_{phase} = \frac{\rho_B \rho_F}{\rho_B + \rho_F} \left( \nabla\phi_B - \frac{1}{2} \nabla\phi_F - A \right)^2 \quad (9)$$

The meaning of this equation is that in a state with paired spinons and condensed bosons at long wavelengths only the combination  $\nabla\phi_B - \frac{1}{2} \nabla\phi_F$  couples to an external vector potential or is relevant to the energy, and the physical superfluid stiffness  $\rho_S = \frac{\rho_B \rho_F}{\rho_B + \rho_F}$ . Similarly one finds

$$H_{mix} = \sum_{\alpha, \sigma} \frac{-\rho_B}{\rho_B + \rho_F} \left( \nabla\phi_B - \frac{1}{2} \nabla\phi_F - A \right) \cdot \vec{v}_1 c_{\alpha\sigma}^+ c_{\alpha\sigma} \quad (10)$$

Because the gauge field has been eliminated, the fermionic degrees of freedom should be regarded not as spinons but as Bogoliubov quasiparticles of the superfluid (or near superfluid) state. They couple only to the combination  $\nabla\phi_B - \frac{1}{2} \nabla\phi_F$  and the coupling is via an effective charge  $Z^e = \frac{-\rho_B}{\rho_B + \rho_F}$  which is negative (hole-like) and vanishes as the Mott insulator is approached.

Eqs 9,10 constitute a derivation, from the  $U(1)$  gauge theory, of phenomenological equations discussed in [18]. The derivation makes it clear that deviations from phenomenological theory occur at length scales shorter than that specified by the scale dependence of the physical superfluid stiffness, i.e. than the shorter of  $\xi_F$  and  $x^{-1/2}$ . The derivation also makes it manifest that the phenomenological action, discussed in [14] on the basis of fermi liquid theory, is more general, and may apply also to situations in which the normal state is not described by fermi liquid theory.

## VORTEX SOLUTION-STATIC CASE

Consider a vortex. Far from the vortex core the fields are found by minimizing  $H_{phase}$  (Eq 9) which implies that  $\nabla^2 (\phi_B - \frac{1}{2} \phi_F - A) = 0$ . Single-valuedness of the wave function implies that both  $\phi_B$  and  $\phi_F$  must have circulation which is an integer multiple of  $2\pi$  so that in a mean field approximation one would write

$$\nabla\phi_B = \frac{m\hat{\theta}}{r} \quad (11)$$

$$\nabla\phi_F = \frac{n\hat{\theta}}{r} \quad (12)$$

The energy associated with a vortex is thus, approximately,

$$E_V = E_{core}(n, m) + \frac{1}{2} \rho_S \left( m - \frac{1}{2} n - A \right)^2 \ln \left( \frac{R}{\xi} \right) \quad (13)$$

where  $\rho_S$  is the physical superfluid stiffness defined below Eq 9,  $R$  is of the order of the inter-vortex separation and  $\xi$  is the length scale below which the supercurrent magnitude deviates from  $1/r$  and will be discussed more fully below.  $E_{core}$  is the core energy of the vortex, i.e. the contribution to the energy arising from scales less than  $\xi$ .

The superflow contribution is clearly minimized by the choice  $m = 0$ ,  $n = 1$ , corresponding to a conventional  $h/2e$  vortex. The core energy term requires more discussion. If  $n = 1$  then a singularity in the fermion pairing amplitude is required. In a clean conventional superconductor one would estimate the energy cost of this singularity as the product of the condensation energy per unit area ( $N_0 \Delta^2$  with  $N_0$  the density of states) and the area of the core ( $\xi_F^2 = v_F^2 / \Delta^2$ ) leading to  $E_{core} \sim v_F^2 N_0$ . In the present problem this implies an  $E_{core}$  of the order of the effective fermi energy  $J$ . On the other hand, if  $n = 0$  and  $m = 1$  then no singularity is required in the fermion field and a calculation very similar to that given in Eqs 19 then shows that the core energy is of the order of the boson or superflow energy  $xt$ , and can be absorbed into the definition of  $\xi$ . These considerations suggest that in the gauge theory the vortex energy may be estimated by

$$E_V \approx C_{core} J (1 - \delta_{n,0}) + C_{sf} x t \ln \left( \frac{R}{\xi} \right) \left( m - \frac{1}{2} n - A \right)^2 \quad (14)$$

with  $C_{core, sf}$  constants. This estimate (proposed and presented in more sophisticated form by Sachdev [2]) suggests that when the superflow energy is dominant (low vortex density or high doping) one has conventional  $h/2e$  vortices but that as  $x$  is reduced or  $R$  is decreased a transition to doubly quantized vortices may occur.

This argument, however, is vitiated by recent work on the structure of the vortex core. From different points of view the authors of [5, 6, 7] show that (within certain reasonable assumptions) some other ordered state, also characterized by an electronic gap of the order of  $\Delta$ , is very nearby in energy and indeed becomes favored as  $x \rightarrow 0$ . The consequence is that  $C_{core}$  in Eq 14 decreases rapidly as  $x \rightarrow 0$  and indeed may even become negative, implying that conventional ( $h/2e$ ) vortices are always favored. In more physical terms, within the classes of models (including the gauge model) considered by [5, 6, 7] the reason that the ground state has superconducting rather than some other sort of order is the gain in energy associated with establishing superfluid phase coherence, so it is natural that even the vortex core energy is set by the phase stiffness. The conventional nature of vortices in this theory was stressed in [16] which however did not consider the core energy explicitly. We note however that even if not favored in the superfluid state, doubly quantized ( $h/e$ ) vortices may be easily excited thermally once the long-ranged superfluid order is disrupted.

We now study the structure of the vortex at shorter

length scales. We consider the case of very weak applied field, so we may choose a gauge in which  $A = 0$ . We assume  $\rho_B \ll \rho_F$  and expand in powers of  $\rho_B$ . We find from Eqs 8, 11

$$a = \frac{1}{2}\nabla\phi_F - \frac{1}{4\rho_F}\rho_B * \nabla\phi_F + \dots \quad (15)$$

where the ellipsis denotes both terms higher order in  $\rho_B/\rho_F$  and fluctuations about the mean field solution for  $\phi_{F,B}$ .

We must now determine the behavior of the boson field and the fermion pairing amplitude. The fermion pairing amplitude varies on the scale  $\xi_F$  which by assumption remains finite as the Mott phase is approached, whereas one expects the bose amplitude to vary on the scale set by the spacing between carriers, which diverges as the Mott phase is approached. We therefore focus on the bose field. In a lightly doped Mott insulator the density of bosons is low. In the limit of dilute bosons one expects [19] that the bose amplitude is described by the two-dimensional Hamiltonian density

$$H_{bose} = \frac{1}{2m_B} ((\nabla - a - A)\psi)^2 + \frac{1}{2}\mu\psi^2 + \frac{1}{4}U\psi^4 \quad (16)$$

In the dilute limit, the parameters  $\mu$  and  $U$  are universal, given in terms of the boson density  $x$  and mass  $m_B$  by

$$\mu = Ux \quad (17)$$

$$U = \frac{4\pi}{m_B \ln(1/x)} \quad (18)$$

We expect that  $m_B \sim 1/tb^2$  with  $b$  the underlying lattice constant. Also, the dilute limit means that a mean field approximation for the boson field is reliable.

We must however consider the mean field approximation for the gauge field in more detail. In the d-wave *RVB* state, fluctuations in  $a$  are controlled by the stiffness corresponding to fermion pairing. This stiffness is large at length scales longer than  $\xi_F$  or energy scales less than the fermion pairing amplitude  $\Delta$  so a mean field approximation is expected to be reliable at long length and low energy scales. However, if  $\xi_F, \Delta^{-1}$  are shorter than the relevant bosonic length scale  $x^{-1/2}$  and time scale  $(xt)^{-1}$  then fluctuations in  $a$  may appreciably renormalize the parameters in  $L_{bose}$ . Because the decay with  $q$  of the fermionic stiffness is slow ( $\sim (\xi_F q)^{-1}$ ) we focus here on the energy scale. As carriers are added to a lightly doped Mott insulator,  $\Delta$  decreases and  $xt$  increases. When  $\Delta/xt$  becomes less than unity we expect that fluctuation corrections to the various parameters become large. We thus distinguish two regimes: a "super-dilute regime" in which  $\Delta > xt$  and  $\gamma \sim 1$  and a "dilute boson regime" in which  $x \ll 1$  but  $\Delta < xt$  and  $\gamma \ll 1$ .

In the limit of interest  $\xi_F < x^{-1/2}$  the boson ground state in the presence of a vector potential is given by the

solution of

$$\frac{-1}{2m} (\nabla + a + A)^2 \psi + U\psi^3 = \mu\psi \quad (19)$$

Eqs 19,12,15 imply that in radial coordinates we have (up to terms of relative order  $x$ )

$$\frac{-1}{2m} \left( \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{4r^2} \right) \psi + U\psi^3 = \mu\psi \quad (20)$$

Defining

$$\psi = x^{1/2} f(r/\xi) \quad (21)$$

with

$$\xi^{-2} = 2m\mu \quad (22)$$

leads to the solution shown in Fig 1. In particular, at large distance  $f \rightarrow 1$  while at small distance

$$f = \sqrt{r} f_0 \quad (23)$$

with  $f_0 = 0.886$ . In other words, as the core of the vortex is approached, the bose amplitude decreases as the square root of the distance from the vortex core.

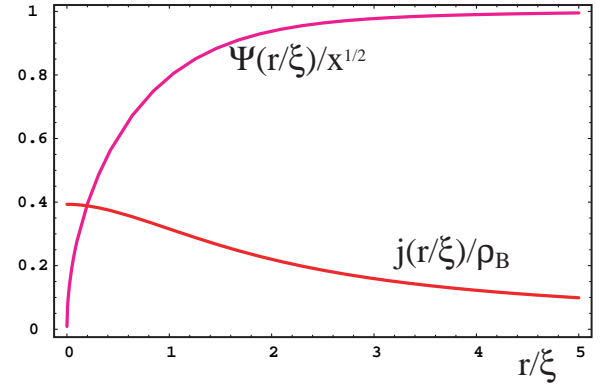


FIG. 1: Variation with distance from vortex core of boson amplitude  $\psi$  and supercurrent  $j$

The supercurrent is given from Eqs 5,16 as

$$j(r) = \frac{|\psi(r)|^2}{m_B} a \quad (24)$$

so in particular at small distances, in physical units

$$j = \frac{f_0}{2\xi} \rho_B(\infty) \quad (25)$$

The resulting current profile is also shown in Fig 1; we see that the supercurrent varies as  $1/r$  for  $r > x^{-1/2}$  and is constant for smaller  $r$ , justifying the qualitative statements made in [16].

The fermion spectrum retains its long-distance value down to a length  $\xi_F = v_1/\Delta$  which is parametrically less

than  $\xi$  as  $x \rightarrow 0$ ; below this length a variety of interesting physical effects [6, 7] may occur. The physical electron spectrum, observable (in principle) via tunnelling, is calculated in the  $U(1)$  gauge theory as a convolution of a holon and a spinon [11], and so is relatively broad in the non-superconducting phase of the model. In the superconducting state the  $q = 0$  boson amplitude develops an expectation value and so the electron spectral function acquires a sharp 'quasiparticle pole' feature. In the limit  $x \rightarrow 0$  the short length scales of the spinon spectrum control the convolution so the coherent part of the spectrum at distance  $r$  from the vortex core is proportional to the boson amplitude at distance  $r$ . In other words, in this theory the strength of the quasiparticle peak measured at a distance  $r$  from the vortex core should begin to decrease for as  $r$  is reduced below  $\xi$ . This effect is not visible in published tunnelling data [21].

To summarize, in the  $U(1)$  gauge theory of a lightly doped Mott insulator, a vortex is characterized by two length scales:  $\xi \sim x^{-1/2}$ , below which the supercurrent ceases to vary as  $1/r$  (and in fact becomes essentially  $r$ -independent) and the scale  $\xi_F$  which does not diverge as the Mott phase is approached and which controls the quasiparticle properties. The core energy is small (of order the superfluid stiffness); however the state in the core possesses a gap very similar to the superconducting gap. This behavior should be contrasted with that of a conventional (BCS) superconductor, in which the length defined by the supercurrent is essentially the same as the length defined by the quasiparticle properties and the core energy is large (of the order of the fermi energy) and the core is gapless apart from the 'finite size effects' which lead to the Caroli-Matricon states.

## MOVING VORTEX

We consider a slowly moving vortex with center position  $\vec{X}_v(t)$ , so that time derivatives of fields may be replaced by the dot product of a field gradient and the vortex velocity; for example  $\partial_t \psi(r, t) = \partial_t \vec{X}_v(t) \cdot \vec{\nabla} \psi$ . The contribution to the action from vortex motion has two terms: a non-dissipative term corresponding to motion in an effective magnetic field  $B_{eff}$  and a dissipative term arising because vortex motion excites fermionic excitations. These terms imply a classical equation of motion

$$B_{eff} \hat{z} \times \partial_t X_V + \eta \partial_t X_V = F_V \quad (26)$$

where  $F_V$  represents the forces acting on the vortex (arising for example from an imposed current and from vortex-vortex interactions). Eq 26 applies only for frequencies less than a cutoff frequency which is the minimum of  $\Delta$  and the boson frequency scale  $x t$ .

$B_{eff}$  may be obtained by considering the action arising from moving a vortex around a closed loop. For orien-

tation we first consider the related purely bosonic problem of a vortex in a two boson condensate. The standard bosonic Lagrangian density [19] includes a 'non-dissipative time derivative' term  $i\gamma_B \psi^\dagger \partial_t \psi$  with coefficient  $\gamma_B = 1$ . A superfluid state is described by a condensate amplitude  $n_s = |\langle \psi \rangle|^2$  and condensate phase  $\phi$ , leading to a term  $i n_s \partial_\tau \phi$  in the action. This term ensures that dragging a long straight vortex in the boson condensate around a loop enclosing an area  $s$  leads to a contribution to the action of  $\Delta S = 2\pi n_s s$ . This contribution is just the action appropriate to a particle in a magnetic field of strength  $h_{eff} = 2\pi n_s$ . For particles on a lattice, a magnetic flux of  $2\pi$  per unit cell has no dynamical consequences, so that one measures  $n_s$  modulo 1 per lattice site. Finally, we consider the relation between the condensate density  $n_s$  and the total boson density  $n_B$ . Gauge invariance means that  $-i\partial_t \phi$  is a chemical potential, and therefore couples to the total particle density,  $n_B$ . In simple boson problems, at  $T = 0$  the only gapless excitation is the phase mode of the superfluid state and therefore  $n_s = n_B$ . As  $T$  is increased from  $T = 0$  gapless 'normal fluid' excitations occur. The presence or absence of Galilean invariance then becomes crucial. One may interpret the  $\partial_t \phi$  term in the vortex action in terms of the acceleration of a vortex in a given force. In a Galilean invariant situation one expects that if a vortex is accelerated it will drag all particles in the system with it, so that the coefficient of  $\partial_t \phi$  is simply the total particle density in the system. However, in a non-Galilean invariant system the superfluid component may accelerate independently of the normal component and  $n_s \leq n_B$ .

We next consider a superconducting condensate made of paired electrons. The important difference here is that 'two-fluid' effects may be important even at very low  $T$  for example because of impurity-induced pairbreaking or of vortices. The conventional result is that if a Landau expansion may be constructed about a non-superconducting state (for example, very near to  $T_c$  or in the presence of strong pair breaking) then one has

$$S_{sc} = \int dt d^d x N_0 T_0^2 \left[ \gamma_F \frac{\Delta + \partial_t \Delta}{T_0^3} + \frac{\xi_0^2 (\nabla \Delta)^2}{T_0^2} + \frac{T - T_c}{T_0} \frac{\Delta^2}{T_0^2} + u \frac{\Delta^4}{T_0^4} \right] \quad (27)$$

Here  $N_0$  is the electronic density of states,  $T_0$  is an energy scale of the order of the transition temperature or the pairbreaking scattering rate  $u$  is a coefficient of the order of unity and the coefficient  $\gamma_F$  of the time derivative term has both real and imaginary parts,  $\gamma_F = \gamma'_F + i\gamma''_F$ . In a usual superconductor the dissipative (real) part of  $\gamma_F$  is of the order of unity (the conventional result is  $\gamma''_F = \pi/8$ ). Because a time dependent phase is a contribution to the chemical potential we may identify the imaginary part  $\gamma''_F$  with  $-\partial T_c / \partial \mu$ . In conventional su-

perconductors this is very small (of order  $T_c/E_F$ ) so the total coefficient of the dissipationless time derivative,  $N_0 T_0 \gamma_F'' \sim (T_c/E_F)^2$  is extremely small and for most purposes may be neglected (for exceptions see e.g [17, 20]). Of course in a conventional superconductor with weak pairbreaking, a Landau expansion only applies for temperatures very near to  $T_c$  and as  $T \rightarrow 0$  one expects  $\gamma_F' \rightarrow 0$  while the dissipationless term  $N_0 T_0 \gamma_F''$  must approach the total fermion density  $n$ . For a type-II superconductor in a magnetic field one similarly expects that  $N_0 T_0 \gamma_F'' \rightarrow n$  only for temperatures of the order of the core state level spacing  $T_c^2/E_F$  and only in the 'super-clean' limit. To summarize, known results from simple fermion and boson problems imply that the dissipationless time derivative term in the superfluid action involves a non-universal coefficient which depends on the interplay between the superfluid and non-superfluid components of the system, and is in general quite small for fermion-based superfluids and is of the order of the particle density for boson-based systems.

We now turn to the boson-fermion-gauge-field problem of interest here. The discussion above shows that there is no simple, generally valid expression for the coefficient, except in the  $T \rightarrow 0$  no-pairbreaking limit, in which the coefficient is the total particle density. Nevertheless a few remarks can be made and limits can be estimated. At high dopings ( $x > J$ ) the bosons condense (or quasi-condense) at temperatures well above  $T_c$  so in this limit superconductivity arises out of a more or less fermi-liquid like state, so near  $T_c$  one expects the resulting superfluid state to be described by a non-dissipative coefficient  $\gamma$  which is of a fermionic order of magnitude and thus much less than unity. We note, though, that in the present model the spinon fermi energy is of order  $J$  and the pairing amplitude varies from a (not too small) fraction of  $J$  at low doping to a very small value at high doping [15] so  $\partial\Delta/\partial\mu$  is not particularly small; the main smallness is provided by the factor  $N_0 T_0 \sim T_c/J$ . On the other hand, as doping is reduced the physics changes. The fermions pair at a higher scale and the superconducting transition is set by the condensation scale of the bosons. An upper bound on the non-dissipative coefficient is then set by the total particle density (modulo 1) i.e.  $\gamma < x$ . In this limit one expects that by temperatures of the order of  $T_c$  the fermions are mostly paired, so that a Landau expansion is not appropriate and the coefficient is set by bosonic physics; i.e.  $\gamma$  is not that different from its  $T = 0$  no-pairbreaking value. We therefore propose the following approximate interpolation formula for the effective magnetic field  $B_{eff}$  implied by the non-dissipative terms in the action of our problem:

$$B_{eff} \approx \frac{1}{(\pi x)^{-1} + (\Delta/J)^{-1}} \quad (28)$$

We turn next to the dissipative term. In the gauge

model, the dissipation arises from the continuum of spinon excitations, and we assume that the temperature, magnetic field or impurity density is large enough that an appreciable number of these exist, and may be characterized by a spinon conductivity  $\sigma_{sp}$  which we take to be local on the scales of interest. The 'electric field' felt by the spinons is  $\partial_t (\frac{1}{2}\nabla\phi - a)$  so that the dissipative contribution to the action is

$$S_{diss} = \frac{\sigma_{sp}}{2} \int d^2 r dt dt' \partial_t \left( \frac{1}{2} \nabla \phi - a \right)_t K(t-t') \partial_{t'} \left( \frac{1}{2} \nabla \phi - a \right)_{t'} \quad (29)$$

with  $K$  the fourier transform of  $1/|\omega|$ . In a slowly moving vortex with center position  $\vec{X}_v(t)$ ,  $\frac{1}{2}\nabla\phi - a$  is a function  $\tilde{a}(\vec{r} - \vec{X}_v(t))$ . Substitution into Eq 29 and some rearrangement leads to the standard form for a particle moving in a dissipative medium, namely

$$S_{diss} = \frac{\eta}{2} \int dt dt' \left( \partial_t \vec{X}_v(t) \right) K(t-t') \left( \partial_{t'} \vec{X}_v(t') \right) \quad (30)$$

with viscosity  $\eta$  given by

$$\eta = \frac{\sigma_{sp}}{2} \int d^2 r \sum_{ij} \left( \frac{\partial (\frac{1}{2}\nabla\phi - a)_i}{\partial r_j} \right)^2 \quad (31)$$

Use of Eq 5 shows that  $\rho_F (\frac{1}{2}\nabla\phi - a)$  is just the physical current  $\vec{j}_{phys} = \rho_B f^2(r/\xi) \hat{\theta}/r$ . Substitution of our result for  $f$  leads to (Note that the logarithm comes from the angular derivative ( $\nabla \rightarrow \frac{1}{r} \partial_\theta$ ))

$$\eta = \frac{\pi \sigma_{sp}}{\xi^2} f_0^2 \left( \frac{\rho_B}{\rho_F} \right)^2 \ln(\xi/\xi_F) \quad (32)$$

We see that as the insulator is approached, the vortex viscosity vanishes very rapidly, indeed as  $x^3$ . One factor of  $x$  comes from the large size of the vortex (proportional to  $\xi^{-2}$ ); the other two factors  $\left( \frac{\rho_B}{\rho_F} \right)^2$  come from the decreased coupling of the vortex motion to the spinons; we interpret this as a signature of the vanishing of the quasi-particle charge in this theory (for a discussion of other signatures see [14, 18]).

It is instructive to view this result in a slightly different way. The 'internal electric field'  $e$  felt by the spinons is  $\partial_t (a - \frac{1}{2}\nabla\phi_F) = \frac{1}{e\rho_F} \partial_t (j_{phys}(r))$ . Application of the usual composition rules of the gauge theory [10] shows that the physical electric field  $E = e\rho_F/\rho_B$  (in the limit  $\rho_F \gg \rho_B$ ) so that the physical electric field  $E$  generated by a moving vortex is

$$\begin{aligned} E(r) &= \frac{1}{e\rho_B} \partial_t (j_{phys}(r)) \\ &= \partial_t (\vec{X}_v(t)) \cdot \vec{\nabla} \left( \frac{f^2(\frac{r-X_V}{\xi}) \hat{\theta}}{r - X_V} \right) \end{aligned} \quad (33)$$

This shows that a moving vortex generates an electric field which varies on the length scale set by the physical current.  $E(r)$  varies as  $1/r^2$  far from the vortex, and as  $1/r$  for  $\xi_F < r < x^{-1/2}$ . A standard argument [22] says that the dissipative contribution may be estimated by multiplying the square of the electric field by the physical conductivity, which from Eq 10 is of order  $(\rho_B/\rho_F)^2$ , leading again to Eq 32 for  $\eta$ .

The result for  $\eta$ , along with the estimates of the core energy, has implications for the width of the resistive superconducting transition in this model. In a two dimensional superconductor, the resistive transition is of the Kosterlitz-Thouless vortex unbinding type. Near to the transition point, but in the normal state, one has a dilute gas of vortices and antivortices. The physical conductivity is the sum of the conductivity due to the moving vortices and the conductivity due to the quasiparticles. Use of Eq 10 and Eq 32 leads to

$$\sigma_{phys} = \left(\frac{\rho_B}{\rho_F}\right)^2 \sigma_{sp} \left(1 + \frac{\pi f_0^2 \ln(\xi_B/\xi_F)}{n_V \xi^2}\right) \quad (34)$$

This, in this model the conductivity is dominated by proximity to the superconducting phase only if vortices are dilute on the length scale set by the supercurrent pattern. However, even when vortices overlap from the 'current' point of view, they may still be dilute on the scale set by the 'quasiparticle core size'  $\xi_F$ , so we would therefore expect the gap in the fermionic excitation spectrum to persist to much higher temperatures than does the 'superfluid' contribution to the conductivity.

The density (and nature) of thermally excited vortices bears further discussion. The factors influencing the density of vortices may be understood from simple mean field free energy arguments. For conventional  $(h/2e)$  vortices the mean-field free energy as a function of conventional vortex density  $n_1$  is (note that the position of the conventional vortex may be defined to within  $\xi_F$  but the logarithm from the superfluid stiffness is cut off by  $\xi$ )

$$E_{n=1,m=0} = E_{core}n_1 - \rho_S n_1 \ln(n_1 \xi^2) - T n_1 \ln(n_1 \xi_F^2) \quad (35)$$

Away from the K-T fluctuation regime,  $\rho_S$  becomes less than  $T$  and we have, approximately,

$$n_1 \xi^2 = \frac{\xi^2}{\xi_F^2} e^{-E_{core}/T} \quad (36)$$

The factor  $\frac{\xi^2}{\xi_F^2}$  makes it easier than one might guess to obtain a reasonable density of conventional vortices; of course the small value of the core energy, arising as discussed above from the proximity in energy of other gapped states within the d-RVB theory, is also important.

It is also of interest to consider the density  $n_2$  of doubly quantized vortices, (i.e. vortices in the boson field only), because their motion leads to dissipation too. As noted above, in this case the core energy is entirely determined by the superfluid stiffness so we have ( $\Phi$  is a scaling function which is exponentially small at large argument and becomes of order unity when its argument becomes of order unity and the factor of 4 comes from the doubly quantized nature of the vortex)

$$n_2 \xi^2 = \Phi(4\rho_S/T) \quad (37)$$

Therefore, even if the core energy of a conventional vortex is very high, (which, in the d-RVB theory it is not) above a temperature scale set by  $\rho_S$  double quantized vortices will proliferate and will suppress the superfluid contribution to the conductivity.

## QUANTAL FLUCTUATIONS AND THE MELTING OF THE VORTEX LATTICE.

We now consider the physics at low temperature in an applied magnetic field. An applied field induces a vortex lattice and, ultimately, a non-superconducting state. In a conventional (BCS) superconductor the transition is driven by the collapse of the superconducting gap. In the present model the important physics involves quantal fluctuations in the positions of the vortices, leading via a first order transition to a 'quantal vortex liquid' state. In particular, although an applied field leads to pairbreaking, and thus to a non-vanishing density of quasiparticles and to a reduction of the superfluid stiffness and gap amplitude, in the gauge model of interest here these effects are small (order  $x^{-2}$ ) [18] so we neglect them.

We estimate the magnitude of the quantal fluctuations of the vortices by considering the fluctuation in position of one vortex about its ideal Abrikosov lattice position. To do this we note that in a vortex lattice the force term in Eq 26 may be written for small amplitude displacements as  $F_V = K_{latt} X_V$  where  $K_{latt}$  is the coefficient of the quadratic term in the restoring potential acting on the vortex and arising from the other vortices in the vortex lattice (plus any pinning forces which may exist, and which we do not treat). Eq 26 applies only for frequencies less than a cutoff frequency which is the minimum of  $\Delta$  and the boson frequency scale  $x t$ .  $K_{latt}$  is of the order of the physical superfluid stiffness  $\rho_B$  divided by the square of the intervortex spacing, i.e.  $K_{latt} = K_0 n_V \rho_B$ . The long range (logarithmic) form of the intervortex potential means that  $K_0 \ll 1$ . We may now quantize Eq 26 and thereby estimate the zero point fluctuations of a vortex as ( $\omega_n$  is a Matsubara frequency and  $\sigma^y$  is a Pauli matrix)

$$\langle X_V^2 \rangle = Tr \sum_n [\eta |\omega_n| + \sigma^y B_{eff} \omega_n + K]^{-1} \quad (38)$$

We estimate that the vortex lattice melts when the mean square vortex displacement divided by the square of the intervortex distance  $b$ , i.e.  $\langle X_V^2 \rangle / b^2$ , becomes of the order of the square of the Lindemann number  $c_L^2$ . The value of the Lindemann number depends on the physical situation and varies from  $c_L^2 = 0.01$  to  $c_L^2 = 0.1$  [26]. The estimate of  $\langle X_v^2 \rangle$  depends on the values of  $\eta$  and  $B_{eff}$  and in particular on whether the system is in the super-dilute regime ( $B_{eff} \sim \pi n_B$  and  $\eta \rightarrow 0$ ) or the dilute regime ( $B_{eff}$  unimportant;  $\eta$  dominant). In the super-dilute regime we neglect the  $\eta$  and find (the numerical factors give the relation between  $b^2$  and  $n_V$  for a triangular vortex lattice)

$$\frac{\langle X_V^2 \rangle}{b^2} = \frac{3\sqrt{3}n_V}{8B_{eff}} \quad (39)$$

In fact, if dissipation is negligible then we may look at the problem in a different way. The vortices move in an effective field which is large, leading to Landau level quantization. If the density of vortices is much less than the density of bosons,  $n_V \ll n_B$  then only the lowest Landau level is populated and we may use known results for the melting of the Wigner crystal in a high magnetic field to argue that lattice melting occurs when  $n_V/n_B \approx 1/10$  corresponding to a Lindemann number  $c_L^2 \approx 0.01$  reasonably consistent with known results for two dimensional triangular lattices [26].

In the dilute regime, the frequency cutoff is needed. If  $\rho_B > \Delta$  then this scale is  $\Delta$  and we obtain

$$\frac{\langle X_V^2 \rangle}{b^2} = \frac{3\sqrt{3}n_V}{4\pi\eta} \ln \left[ 1 + \frac{\Delta\eta}{K} \right] \quad (40)$$

Use of Eq 32 and our estimate for  $K$  shows that the important dimensionless parameter is given by

$$\frac{\Delta\eta}{K} \sim \frac{\Delta}{\rho_B} \left( \frac{\xi_V^2}{\xi_B^2} \right) \frac{\pi\sigma_{sp}}{K_0} \left( \frac{\rho_B}{\rho_F} \right)^2 \ln \left( \frac{\xi_B}{\xi_F} \right) \quad (41)$$

This parameter may be larger or smaller than unity, because  $\rho_B/\rho_S \sim x$  while  $K_0 \ll 1$  and  $\sigma_{sp}$  must by consistency be of order  $1/x$  (because we have written clean-limit formulae which require that  $\sigma_{sp} = p_F l_F / 2\pi > p_F \xi_F / 2\pi > \rho_S / \rho_B \sim 1/x$ ). However, as one goes more deeply into the small  $\Delta$  regime the parameter shrinks and we obtain

$$\frac{\langle X_V^2 \rangle}{b^2} = \frac{3\sqrt{3}\Delta}{4\pi K_0 \rho_B} \left( \frac{\xi_V^2}{\xi_B^2} \right) \quad (42)$$

Thus in the dilute limit the lattice still melts when the flux per boson is of the order of the square of the Lindemann number, up to a factor of order  $\rho_B/\Delta$ .

The melted phase is an interesting example of a 'non-fermi-liquid': the vortex motion is damped (albeit weakly) so the model is characterized by a non-vanishing  $\sigma_{xx}$  and  $\sigma_{xy}$ , as in a normal metal, (although the  $\sigma_{xy}$

value is rather large) but also by a 'non-Luttinger' fermi surface with a gap over large regions of the nominal fermi surface.

## CONCLUSION

One of the most interesting aspects of high temperature superconductivity is the 'pseudogap' regime of underdoped materials. This regime is characterized by a gap (of approximately d-wave form) in the quasiparticle spectrum but neither long ranged superconducting order nor particularly noticeable superconducting fluctuations. One possibility is that this regime involves electron pairing (as in a conventional superconductor) but with long ranged superconducting order disrupted by strong phase fluctuations, arising physically from the strong suppression of charge response expected near a Mott insulator. A difficulty with this idea is the absence of noticeable 'paraconductivity': transport measurements indicate a critical regime of order  $10K$  at most [23, 24].

In this paper we studied one theoretical implementation of the 'phase fluctuation' scenario for the pseudogap, namely the d-RVB regime of the  $U(1)$  gauge theory of lightly doped Mott insulators. In this approach the spin degrees of freedom are mostly paired into a d-wave pairing state and the low  $T$  charge response is essentially that of a superfluid, but with properties strongly affected by proximity to the Mott transition. One important feature of the model is that the vortex excitations are characterized by two length scales: the 'quasiparticle coherence length'  $\xi_F = v/\Delta$  which controls the distance over which the excitation spectrum differs from that far from a vortex, and the 'current coherence length'  $\xi$  which varies as the square root of the doping and cuts off the familiar  $1/r$  divergence of the supercurrent near a vortex. We studied the charge transport properties (many of which are dominated by vortices) and showed in particular that the electric field created by a moving vortex, the dissipation due to moving vortex, the value of  $H_{c2}$  and the size of the fluctuation regime near the resistive transition are all controlled by  $\xi$  which diverges near the Mott transition, rather than by the 'quasiparticle length'  $\xi_F$  which does not.

As noted elsewhere [8, 14, 18] the theory disagrees in a number of ways with experiments; the most significant difficulty is the small value and strong doping dependence of the 'quasiparticle charge' defined in Eq 10. Further, our calculation has a number of phenomenological aspects. For example, we assumed a finite 'spinon conductivity' which could reasonably be expected to arise from the 'gapless fermi arcs' induced by a non-vanishing temperature or applied magnetic field, but we did not attempt to calculate this from first principles, nor did we investigate the subtle quantum mechanics of fermions in the presence of conventional vortices. How-



ever, we believe the results presented here are useful because they provide an explicit demonstration in a well-defined model that if the supercurrent-defined correlation length is parametrically larger than the quasiparticle-defined length, then the resistive properties are controlled by the length scale over which the supercurrent varies.

Other workers have observed that the theory admits doubly quantized vortices. We have noted that they proliferate above a scale defined by the physical superfluid stiffness and (if the core energy of conventional vortices were large, which it is not in this model) would suppress the superconducting fluctuation contribution to the conductivity. We also showed how, in this model, the 'non-dissipative time derivative', whose importance was stressed in [17], is important for the estimation of the upper critical field, and argued that it crosses over from the 'fermionic' value  $T_c/E_F$  to the 'bosonic' value proportional to the density of charges  $x$ . A subsequent paper will apply these ideas to a different model of high- $T_c$  superconductivity.

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## APPENDIX: FERMI-LIQUID-BASED APPROACHES TO DOPED MOTT INSULATORS

This Appendix treats the case of superconductivity developing out of a state which is well described by the usual fermi liquid theory. The necessary formalism was developed by Larkin and Leggett, and some of the results were sketched elsewhere [25]. A fermi-liquid state is characterized by a quasiparticle velocity  $v^*(\theta)$  which may depend on position ( $\theta$ ) on the fermi surface, a characteristic energy scale  $E^*$  and a Landau interaction function  $T(\theta, \theta')$ . In order for fermi liquid theory to be applicable, the maximum superconducting gap  $\Delta_0$  must be less than  $E^*$ . Transcription of the standard results to the language of the section above leads to

$$\rho_{s0} = \langle v^*(\theta) (1 - T)_{\theta, \theta'}^{-1} v^*(\theta') \rangle \quad (43)$$

$$v_F = v^* \quad (44)$$

$$\frac{d\rho_S}{dT} = \frac{\langle v^*(\theta) (1 - T)_{\theta, \theta_1}^{-1} L(\theta_1) (1 - T)_{\theta_1, \theta'}^{-1} v^*(\theta') \rangle}{\langle v^*(\theta)^2 \rangle} \quad (45)$$

where the angle bracket means multiplication by density of states and average over the fermi line and  $L(\theta) = \int \frac{d\omega d\theta}{2\pi^2} \frac{\omega}{2T^2 \coth(\frac{\omega}{2T})} \frac{\Delta(\theta)^2}{(\omega^2 + \Delta(\theta)^2)^{3/2}}$ .

Comparison of these results with those presented in section II shows that the  $U(1)$  theory corresponds to a fermi liquid with a weakly angle-dependent Landau interaction function whose 'current-channel' value is of order  $1/x$  [25].

We now consider the situation in more detail by calculating the low- $T$  current-current correlation function for a superconducting fermi liquid, making the usual assumption that the Landau interaction function may be decomposed into angular channels in the conventional way, and that the maximum value of the superconducting gap is small compared to the characteristic quasiparticle energy scale  $E^*$  so that quasiparticle damping effects may be neglected. The gauge-invariant current-current correlation function is then [25]

$$\chi_{jj}(q, \Delta) = \frac{\chi_{qp}(q, \Delta)}{1 + I_1 \chi_{qp}(q, \Delta)} \quad (46)$$

with  $I_1$  the current-channel Landau interaction parameter (so that the Landau parameter  $F_{1s} = 2I_1 \chi_{qp}(0, \Delta)$ ) and

$$\chi_{qp}(q, \Delta) = T \sum_{\omega} \int \frac{d\theta}{(2\pi)^2 v^*(\theta)} v_x^{*2} \frac{\Delta(\theta)^2}{\sqrt{\Delta(\theta)^2 + \omega^2}} \frac{1}{\left( \omega^2 + \Delta(\theta)^2 + (v^* q \cos(\theta - \theta_q))^2 / 4 \right)} \quad (47)$$

(Note that the result depends on the angle  $\theta_q$  between the direction of  $q$  and the nodes in the gap).

Standard calculations [27] show that at  $vq \gg \Delta_0$ ,  $\chi_{qp} \sim \Delta_0/vq$  so that we expect an appreciable change in  $\chi_{jj}$  when  $vq \sim \Delta_0/x$ . In other words, a naive application of fermi liquid theory would predict a very short characteristic length scale of order  $xv/\Delta$ . However, from the usual physical picture of the doped Mott insulator as a dilute collection of holes one might expect the interparticle spacing  $x^{-1/2}$  to be an important scale. The discrepancy is resolved by noting that the Landau parameter presumably varies on the scale  $x^{-1/2}$ . This effect is beyond the scope of Landau theory but is captured in the  $U(1)$  approach.

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- [1] For a review, see J. Orenstein and A. J. Millis, *Science* **288** 468-74 (2000).
  - [2] S. Sachdev *Phys. Rev.* **B45**, 389-399 (1992)
  - [3] C. Wynn, D.A. Bonn, B.W. Gardner, Yu-Ju Lin, Ruixing Liang, W.N. Hardy, J.R. Kirtley, and K.A. Moler, *Phys. Rev. Lett.* in press.
  - [4] See, e.g. Ch. 6 of V. N. Kopnin, *Theory of Nonequilibrium Superconductivity* (Clarendon Press, Oxford, 2001).
  - [5] D. P. Arovas, A. J. Berlinsky, C. Kallin, and S.-C. Zhang, *Phys. Rev. Lett.* **79**, 2871-2874 (1997).
  - [6] Jung Hoon Han, Dung-Hai Lee, *Phys. Rev. Lett.* **85**, 1100-3 (2000).
  - [7] Jun-ichiro Kishine, Patrick A Lee and Xiao-Gang Wen, *Phys. Rev. Lett.* **86**, 5365 (2001).
  - [8] P. A. Lee and X.-G. Wen *Phys. Rev. Lett.* **77**, 4111 (1997).
  - [9] V. Geshkenbein, L. B. Ioffe and A. J. Millis, *Phys. Rev. Lett.* **80** 5778-81 (1998).

- [10] L. B. Ioffe and A. I. Larkin, Phys. Rev. **B 39**, 8988-8999 (1989).
- [11] P. W. Anderson, Science **235** 1196 (1987).
- [12] N. Nagaosa and P. A. Lee, Phys. Rev. Lett. **64**, 2450-2453 (1990)
- [13] M. Grilli, B. G. Kotliar and A. J. Millis, Physical Review **B42**, 329-41, (1990).
- [14] A. J. Millis, S. M. Girvin, L. B. Ioffe and A. I. Larkin, J. Phys. Chem. Solids **59**, 1742-5 (1998).
- [15] B.G. Kotliar and J. Liu, Phys. Rev **B38**, 5142 (1988); Y. Suzumura et. al. J. Phys. Soc Jpn **57** 2768-72 (1988).
- [16] D. H. Lee, Phys. Rev. Lett. **84** 2694-7 (2000).
- [17] V. B. Geshkenbein, L. B. Ioffe, and A. I. Larkin, Phys. Rev. B **55**, 3173-3180 (1997).
- [18] L. B. Ioffe and A. J. Millis, cond-mat/0112509
- [19] V. N. Popov, *Collective Excitations and Functional Integrals* (Cambridge University Press, Cambridge, 1987).
- [20] A. T. Dorsey and S. Ullah, Phys. Rev. **B44** 262-73 (1991).
- [21] S. Pan et. al., Phys. Rev. Lett. **85** 1536-9 (2000).
- [22] see Sec 5.5.1 of M. Tinkham *Introduction to Superconductivity*, (2nd, ed, McGraw Hill, New York, 1996).
- [23] J. Corson et al., Nature **398**, 221 (1999).
- [24] J. M. Harris, Y. F. Yan, P. Matl, N. P. Ong, P. W. Anderson, T. Kimura and K. Kitazawa, Phys. Rev. Lett. **75** 1391-4 (1995).
- [25] A. I. Larkin, Sov. Phys. JETP **14** 1498 (1964); A. J. Leggett, Phys. Rev. **140**, A1869-A1888 (1965).
- [26] See, e.g. H. Nordborg and G. Blatter, Phys. Rev. **58**, 14556 (1998) (2+1 dim vortex lattice, scalar and vector interactions  $c_L \approx 0.25$ ); S. Ryu, S. Doniach, Guy Deutscher, and A. Kapitulnik, Phys. Rev. Lett. **68**, 710 (1992) (simulation of superconductor in magnetic field  $c_L \approx 0.1$ ); L. Xing and Z. Tesanovich, Phys. Rev. Lett. **65**, 794 (1990) (2+1 dim vortex lattice, scalar interaction only  $c_L \approx 0.28$ ); Ph. Choquard and J. Clerouin, Phys. Rev. Lett. **50**, 2086 (1983); V. M. Bedanov et al, ZhETF **88**, 1622 (1985) JETP **61**, 967 (1985) (thermal melting of 2d Wigner crystal with coulomb and short ranged interactions; shown in M.V. Feigelman, V.B. Geshkenbein, L.B. Ioffe, A.I. Larkin, Phys. Rev. B, **48**, 16641 (1993) to be equivalent to  $c_L \approx 0.08$ .)
- [27] A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics*, (Dover, New York: 1964).